

# Explicit determination of mean first-passage time for random walks on deterministic uniform recursive trees

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The determination of mean first-passage time (MFPT) for random walks in networks is a theoretical challenge, and is a topic of considerable recent interest within the physics community. In this paper, according to the known connections between MFPT, effective resistance, and the eigenvalues of graph Laplacian, we first study analytically the MFPT between all node pairs of a class of growing treelike networks, which we term deterministic uniform recursive trees (DURTs), since one of its particular cases is a deterministic version of the famous uniform recursive tree. The interesting quantity is determined exactly through the recursive relation of the Laplacian spectra obtained from the special construction of DURTs. The analytical result shows that the MFPT between all couples of nodes in DURTs varies as  $N \ln N$  for large networks with node number  $N$ . Second, we study trapping on a particular network of DURTs, focusing on a special case with the immobile trap positioned at a node having largest degree. We determine exactly the average trapping time (ATT) that is defined as the average of FPT from all nodes to the trap. In contrast to the scaling of the MFPT, the leading behavior of ATT is a linear function of  $N$ . Interestingly, we show that the behavior for ATT of the trapping problem is related to the trapping location, which is in comparison with the phenomenon of trapping on fractal T-graph although both networks exhibit tree structure. Finally, we believe that the methods could open the way to exactly calculate the MFPT and ATT in a wide range of deterministic media.

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## I. INTRODUCTION

The field of complex networks has been very active in the past decade, since they have been proven a powerful tool to describe very diverse systems in nature and society [1–5]. One of the ultimate goals of the study of complex networks is to understand the influences of network structure on dynamics running on them [3–5]. Among various dynamical processes, random walks on networks are fundamental to many branches of science and engineering and have received a surge of interest in recent years [6–13]. As a basic dynamical process, random walks are relevant to a variety of aspects of complex networks, such as target problem [14], community detection [15,16], network routing [17], reaction-diffusion processes [18,19], and so on. Therefore, it is of major theoretical interest and practical importance to investigate random walks on complex networks.

A primary quantity of interest that relates to random walks is the first-passage time (FPT) defined as the expected time for a walker to first reach a given destination node starting from a source point. The importance of FPTs originates from the following main aspects. First, their first encounter properties are relevant to those in a plethora of real situations [11], such as transport, disease spreading, target search, to name a few. On the other hand, they can measure the efficiency of random navigation on networks [20]. Last but not least, many other quantities for random walks can be ex-

pressed in terms of FPTs, and much information about the random-walks dynamics is encoded in FPTs [21]. Recently, there have been a growing number of theoretical studies on FPTs [6,9,11,13]. Many authors have devoted their endeavors to study the average of FPTs to hit a given target node from all other nodes [22–29]. In addition, relevant work also addressed the FPTs between all couples of nodes, giving some numerical results [30,31] or approximate scalings [9–12,32–34]. These simulation results and scaling laws are necessary as a first step toward understanding random walks on networks; however they do not provide a complete picture of the random-walk dynamics, and analytical exact solutions are helpful in this regard [35].

In this paper, we study analytically random walks on a class of deterministic treelike networks. By using the links between the random walks, electrical networks, and Laplacian spectra, we first compute exactly the mean first-passage time (MFPT) between two nodes over all pairs of nodes. The obtained explicit formula indicates that for large networks with  $N$  nodes, the MFPT is asymptotic to  $N \ln N$ . Then, we study the trapping problem, a particular random-walk issue, on a special case of the network family with a trap fixed at a node of the highest degree. We derive rigorously the average tapping time (ATT), which is the average of FPTs from all nodes to the trap. We show that in contrast to the scaling of MFPT, the leading behavior of ATT grows lineally with  $N$ . Since the MFPT can be considered as the average of ATT with the trap distributed uniformly on all nodes of the entire network, we conclude that the trap location has an important influence on the behavior of ATT. We expect that the our analysis technique could be applicable to determining MFPT and ATT for a broad range of deterministic networks.

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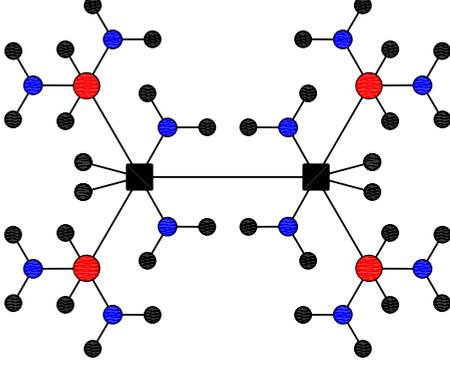


FIG. 1. (Color online) The first several iterative processes of a particular network for the case of  $m=2$ .

## II. DETERMINISTIC UNIFORM RECURSIVE TREES

We first introduce the model concerned, which is a class of trees (networks) defined in an iterative method [36]. Let  $U_g (g \geq 0)$  denote the networks after  $g$  iterations. Then the networks can be generated as follows. Initially ( $g=0$ ),  $U_0$  has two nodes connected by an edge. For  $g \geq 1$ , we can obtain  $U_g$  from  $U_{g-1}$  by adding  $m$  ( $m$  is a natural number) new offspring nodes to each existing node in  $U_{g-1}$ . At each iteration  $g_i (g_i \geq 1)$ , the number of newly generated nodes is  $L(g_i) = 2m(m+1)^{g_i-1}$ . Thus, the order (i.e., number of nodes) and the number of edges in  $U_g$  are  $N_g = \sum_{g_i=0}^g L(g_i) = 2(m+1)^g$  and  $K_g = N_g - 1 = 2(m+1)^g - 1$ , respectively. Figure 1 shows an example of the network family for a special case of  $m=2$  after four iterations.

Note that the network for the special case of  $m=1$  is actually a deterministic version of the uniform recursive tree (URT) [37], which is a principal famous model [38,39] for random graphs [40] and has a variety of important applications in many aspects [41–43]. Moreover, since this particular case of the networks under consideration has similar topological characteristics as the URT, we call the investigated networks *deterministic uniform recursive trees* (DURTs), which could be helpful for better understanding of the nature of the URT.

We study this model because of its intrinsic interest [44–48] and its relevance to real-life networks. For instance, it is small world [36,44,46,47,49]; particularly, the so-called border tree motifs have been shown to be present, in a significant way, in real-world systems [50]. In the rest of this paper, we will study random walks performed on DURTs with an aim to better understand dynamical process occurring on them.

## III. FORMULATING STANDARD RANDOM-WALKS ON DURTS

We study a simple model for random walks on the DURTs  $U_g$ . At each time step, the walker makes a jump from its current location to any of its nearest neighbors with uniform probability. We are interested in the FPT of a random walker starting from a source to a given target point, averaged over all node pairs of source and target points.

To determine the FPT between a pair of two different nodes, one can make use of the method of the pseudoinverse of the Laplacian matrix [51] for  $U_g$ , where random walks are performed. It is a very efficient method, which allows to obtain the FPT between two arbitrarily distinct nodes directly from the network topology and only requires inversion of a single  $N_g \times N_g$  matrix. The pseudoinverse of the Laplacian matrix,  $\mathbf{L}_g^\dagger$ , of  $U_g$ , is in fact a variant of the inverse of its Laplacian matrix,  $\mathbf{L}_g$ . The elements  $l_{ij}^g$  of the latter are defined as follows:  $l_{ij}^g = -1$  if nodes  $i$  and  $j$  are connected by a link, otherwise  $l_{ij}^g = 0$ ; while  $l_{ii}^g = k_i$  (viz., degree of node  $i$ ). Then, the pseudoinverse of the Laplacian matrix  $\mathbf{L}_g$  is defined to be [52]

$$\mathbf{L}_g^\dagger = \left( \mathbf{L}_g - \frac{\mathbf{e}_g \mathbf{e}_g^\top}{N_g} \right)^{-1} + \frac{\mathbf{e}_g \mathbf{e}_g^\top}{N_g}, \quad (1)$$

where  $\mathbf{e}_g$  is the  $N_g$ -dimensional “one” vector, i.e.,  $\mathbf{e}_g = (1, 1, \dots, 1)^\top$ .

We use  $F_{ij}(g)$  to denote the FPT for the walker in  $U_g$ , starting from node  $i$  to node  $j$ , which can be expressed in terms of the entries  $l_{ij}^{\dagger,g}$  of  $\mathbf{L}_g^\dagger$  as follows [31]:

$$F_{ij}(g) = \sum_{n=1}^{N_g} (l_{in}^{\dagger,g} - l_{ij}^{\dagger,g} - l_{jn}^{\dagger,g} + l_{jj}^{\dagger,g}) l_{nn}^g, \quad (2)$$

where  $l_{nn}^g$  is the  $n$  entry of the diagonal of the Laplacian matrix  $\mathbf{L}_g$ . So the total,  $F_{\text{tot}}(g)$ , for FPTs between all pairs of nodes in  $U_g$  reads

$$F_{\text{tot}}(g) = \sum_{i \neq j} \sum_{j=1}^{N_g} F_{ij}(g), \quad (3)$$

and the MFPT averaged over all node pairs,  $\langle F \rangle_g$ , is then

$$\langle F \rangle_g = \frac{F_{\text{tot}}(g)}{N_g(N_g - 1)} = \frac{1}{N_g(N_g - 1)} \sum_{i \neq j} \sum_{j=1}^{N_g} F_{ij}(g). \quad (4)$$

Equations (2) and (4) show that the issue of computing  $\langle F \rangle_g$  is reduced to finding the elements of the pseudoinverse matrix  $\mathbf{L}_g^\dagger$ . Since for large  $g$  the network order  $N_g$  increases exponentially with  $g$ , it becomes intractable to obtain  $\langle F \rangle_g$  through direct calculation using the pseudoinverse matrix because of the limitations of time and computer memory, and one can compute directly the MFPT only for the first iterations (see Fig. 2). Thus, it would be satisfactory if good methods could be proposed to get around this problem. Fortunately, the particular construction of the DURTs and the connection [53,54] between effective resistance and the FPTs for random walks allow us to calculate analytically the MFPT to obtain a rigorous solution. Details will be provided below.

## IV. EXACT SOLUTION TO MFPT AVERAGED OVER ALL NODE PAIRS

In order to avoid the computational complexity of inverting the matrix, in what follows, we will use the connection between the electronic networks and random walks to find a closed-form formula for MFPT,  $\langle F \rangle_g$ .

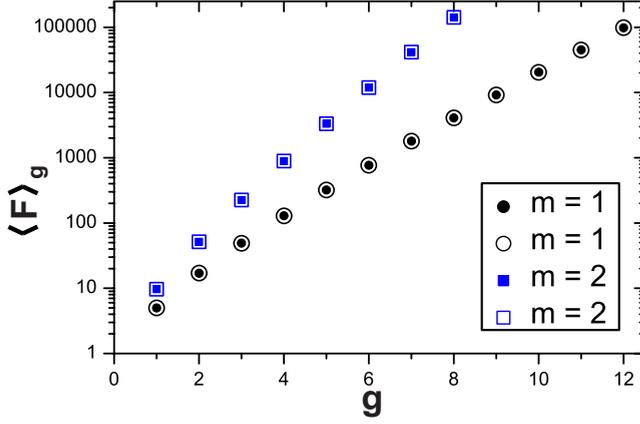


FIG. 2. (Color online) Mean first-passage time  $\langle F \rangle_g$  as a function of the iteration  $g$  on a semilogarithmic scale for different values of parameter  $m$ . The filled symbols are the numerical results obtained by direct calculation from Eqs. (2) and (4), while the empty symbols correspond to the analytical values from Eq. (25), both of which are consistent with each other.

#### A. Relation for commute time and effective resistance between two nodes

For a given graph  $G$ , its underlying electrical network [55] can be obtained by replacing each edge of  $G$  with a unit resistor. The effective resistance of electrical network provides an alternative way to compute FPTs for random walks on the original network [53,54]. It has been proven that for a connected graph, the FPTs,  $F_{ij}$  and  $F_{ji}$  between nodes  $i$  and  $j$ , and the effective resistance,  $R_{ij}$ , between these two nodes satisfy the following relation:

$$F_{ij} + F_{ji} = 2KR_{ij}, \quad (5)$$

where  $K$  is the number of all edges in the graph and  $F_{ij}$  is the expected time that a random walker spends on reaching node  $j$  for the first time, starting from node  $i$ . Actually, the sum,  $F_{ij} + F_{ji}$ , is the average time for a walker to go from  $i$  to  $j$  and back or vice versa, and it is often called commute time [56] represented by  $C_{ij}$ , i.e.,  $C_{ij} = F_{ij} + F_{ji}$ . By symmetry,  $C_{ij} = C_{ji}$ . Then, Eq. (5) can be recast as

$$C_{ij} = C_{ji} = 2KR_{ij}. \quad (6)$$

Thus, if we view  $U_g$  as resistor networks by considering each edge to be unit resistor, according to the close relation between FPTs and effective resistance shown in Eqs. (5) and (6), Eq. (3) can be rewritten as

$$F_{\text{tot}}(g) = \frac{1}{2} \sum_{i \neq j} \sum_{j=1}^{N_g} C_{ij}(g) = K_g \sum_{i \neq j} \sum_{j=1}^{N_g} R_{ij}(g), \quad (7)$$

where  $C_{ij}(g)$  and  $R_{ij}(g)$  represent, respectively, the commute time and effective resistance between two nodes  $i$  and  $j$  of  $U_g$ . Analogously, Eq. (4) can be recast in terms of effective resistances as

$$\langle F \rangle_g = \frac{F_{\text{tot}}(g)}{N_g(N_g - 1)} = \frac{1}{N_g} \sum_{i \neq j} \sum_{j=1}^{N_g} R_{ij}(g), \quad (8)$$

where the sum of effective resistors between all pairs of nodes of  $U_g$  is the so-called Kirchhoff index [57], which we denote by  $R_{\text{tot}}(g)$ . Using the previously obtained results [58,59], the following relation holds:

$$R_{\text{tot}}(g) = \sum_{i \neq j} \sum_{j=1}^{N_g} R_{ij}(g) = 2N_g \sum_{i=2}^{N_g} \frac{1}{\lambda_i}, \quad (9)$$

where  $\lambda_i$  ( $i=2, \dots, N_g$ ) are all the nonzero eigenvalues of Laplacian matrix,  $\mathbf{L}_g$ , of network  $U_g$ . Note that since  $U_g$  is connected, its Laplacian matrix has only one zero eigenvalue  $\lambda_1$ , i.e.,  $\lambda_1=0$ . Then, we have

$$\langle F \rangle_g = 2 \sum_{i=2}^{N_g} \frac{1}{\lambda_i}. \quad (10)$$

Having  $\langle F \rangle_g$  in terms of the sum of the reciprocal of all nonzero Laplacian eigenvalues, the next step is to determine this sum.

#### B. Determining MFPT using Laplacian eigenvalues

After reducing the problem to finding the total of the reciprocal of all nonzero eigenvalues of  $\mathbf{L}_g$ , in the following text, we will resolve this problem.

By construction, it is easy to derive the following recursion relation between  $\mathbf{L}_g$  and  $\mathbf{L}_{g-1}$ ,

$$\mathbf{L}_g = \begin{pmatrix} \mathbf{L}_{g-1} + m\mathbf{I} & -\mathbf{I} & -\mathbf{I} & \cdots & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{pmatrix}, \quad (11)$$

where each block is a  $2(m+1)^{t-1} \times 2(m+1)^{t-1}$  matrix and  $\mathbf{I}$  is the identity matrix. Then, using the elementary matrix operations and the results in [60], the characteristic polynomial,  $P_g(x)$ , of  $\mathbf{L}_g$  satisfies [48]

$$\begin{aligned} P_g(x) &= \det(x\mathbf{I} - \mathbf{L}_g) \\ &= \{\det[(x-1)\mathbf{I}]\}^m \det \left[ \left( x - m - \frac{m}{x-1} \right) \mathbf{I} - \mathbf{L}_{g-1} \right], \end{aligned} \quad (12)$$

which can be rewritten recursively as

$$P_g(x) = (x-1)^{2m(m+1)^{g-1}} P_{g-1}[\varphi(x)], \quad (13)$$

where

$$\varphi(x) = x - m - \frac{m}{x-1}. \quad (14)$$

Since there are  $2(m+1)^g$  nodes in  $U_g$ , the Laplacian matrix  $\mathbf{L}_g$  has  $2(m+1)^g$  eigenvalues, which are represented as  $\lambda_1^g, \lambda_2^g, \dots, \lambda_{2(m+1)^g}^g$ , respectively. We denote by  $E_g$  the set of these Laplacian eigenvalues, i.e.,  $E_g = \{\lambda_1^g, \lambda_2^g, \dots, \lambda_{2(m+1)^g}^g\}$ ,

and we assume that  $\lambda_1^g \leq \lambda_2^g \leq \dots \leq \lambda_{2(m+1)^g}^g$ . The set  $E_g$  can be classified into two subsets represented by  $E_g^{(1)}$  and  $E_g^{(2)}$ , respectively [48]. That is to say,  $E_g = E_g^{(1)} \cup E_g^{(2)}$ , where  $E_g^{(1)}$  consists of eigenvalue 1 with multiplicity  $2(m-1)(m+1)^{g-1}$ ,

$$E_g^{(1)} = \underbrace{\{1, 1, 1, \dots, 1, 1\}}_{2(m-1)(m+1)^{g-1}}, \quad (15)$$

in which the distinctness of elements has been ignored.

The remaining  $4(m+1)^{g-1}$  eigenvalues of  $L_g$ , forming the subset  $E_g^{(2)}$ , are determined by equation  $P_{g-1}[\varphi(x)] = 0$  and expressed separately by  $\bar{\lambda}_1^g, \bar{\lambda}_2^g, \dots, \bar{\lambda}_{4(m+1)^{g-1}}^g$ . For the sake of convenience, we presume  $\bar{\lambda}_1^g \leq \bar{\lambda}_2^g \leq \dots \leq \bar{\lambda}_{4(m+1)^{g-1}}^g$ . Thus,  $E_g^{(2)} = \{\bar{\lambda}_1^g, \bar{\lambda}_2^g, \dots, \bar{\lambda}_{4(m+1)^{g-1}}^g\}$ .

According to Eq. (14), it is obvious that for an arbitrary element in  $E_{g-1}$ , say  $\lambda_i^{g-1} \in E_{g-1}$ , both solutions of  $x - m - \frac{m}{x-1} = \lambda_i^{g-1}$  belong to  $E_g^{(2)}$ . To facilitate the following computation, we rewrite equation  $x - m - \frac{m}{x-1} = \lambda_i^{g-1}$  in an alternative way as

$$x^2 - (\lambda_i^{g-1} + m + 1)x + \lambda_i^{g-1} = 0. \quad (16)$$

Moreover, we use notations  $\bar{\lambda}_i^g$  and  $\bar{\lambda}_{i+2(m+1)^{g-1}}^g$  to represent it two solutions, which provide a natural increasing order of the Laplacian eigenvalues of  $U_g$  [48]. Solving Eq. (16), its two roots are obtained to be

$$\bar{\lambda}_i^g = \frac{1}{2}[\lambda_i^{g-1} + m + 1 - \sqrt{(\lambda_i^{g-1} + m + 1)^2 - 4\lambda_i^{g-1}}], \quad (17)$$

and

$$\bar{\lambda}_{i+2(m+1)^{g-1}}^g = \frac{1}{2}[\lambda_i^{g-1} + m + 1 + \sqrt{(\lambda_i^{g-1} + m + 1)^2 - 4\lambda_i^{g-1}}], \quad (18)$$

respectively. Inserting each of the  $2(m+1)^{g-1}$  elements of  $E_{g-1}$  into Eqs. (17) and (18), one obtains the subset  $E_g^{(2)}$  with cardinality  $4(m+1)^{g-1}$ . Considering  $E_0 = \{0, 2\}$  and recursively applying Eqs. (17) and (18), all Laplacian eigenvalues of  $U_g$  can be fully determined.

Having obtaining the recursive solutions of the Laplacian spectra of  $U_g$ , we continue to calculate the sum on the right-hand side of Eq. (10), which is represented by  $S_g$  henceforth. Note that although we fail to determine all the eigenvalues of  $L_g$  in an explicit way, we will show that it is possible to provide a closed-form expression for  $S_g$ . By definition, we have

$$S_g = \sum_{i=2}^{2(m+1)^g} \frac{1}{\lambda_i^g} = \sum_{\lambda_i^g \in E_g^{(1)}} \frac{1}{\lambda_i^g} + \sum_{i=2}^{4(m+1)^{g-1}} \frac{1}{\bar{\lambda}_i^g}. \quad (19)$$

we denote the two sums by  $S_g^{(1)}$ , and  $S_g^{(2)}$ , respectively. From Eq. (15), we can easily get the first sum,

$$S_g^{(1)} = 2(m-1)(m+1)^{g-1}. \quad (20)$$

The second sum can be evaluated as

$$\begin{aligned} S_g^{(2)} &= \sum_{i=2}^{4(m+1)^{g-1}} \frac{1}{\bar{\lambda}_i^g} \\ &= \sum_{i=2}^{2(m+1)^{g-1}} \left( \frac{1}{\bar{\lambda}_i^g} + \frac{1}{\bar{\lambda}_{i+2(m+1)^{g-1}}^g} \right) + \frac{1}{\bar{\lambda}_{1+2(m+1)^{g-1}}^g} \\ &= \sum_{i=2}^{2(m+1)^{g-1}} \frac{\bar{\lambda}_i^g + \bar{\lambda}_{i+2(m+1)^{g-1}}^g}{\bar{\lambda}_i^g \bar{\lambda}_{i+2(m+1)^{g-1}}^g} + \frac{1}{\bar{\lambda}_{1+2(m+1)^{g-1}}^g}. \end{aligned} \quad (21)$$

Since  $\bar{\lambda}_i^g$  and  $\bar{\lambda}_{i+2(m+1)^{g-1}}^g$  are the two roots of Eq. (16), according to the Vieta's formulas, we have  $\bar{\lambda}_i^g + \bar{\lambda}_{i+2(m+1)^{g-1}}^g = \lambda_i^{g-1} + m + 1$  and  $\bar{\lambda}_i^g \bar{\lambda}_{i+2(m+1)^{g-1}}^g = \lambda_i^{g-1}$ . Moreover, considering  $\lambda_1^g = 0$ , so  $\lambda_{1+2(m+1)^{g-1}}^g = m + 1$ . Then, Eq. (21) is rewritten as

$$\begin{aligned} S_g^{(2)} &= \sum_{i=2}^{2(m+1)^{g-1}} \frac{\lambda_i^{g-1} + m + 1}{\lambda_i^{g-1}} + \frac{1}{m + 1} \\ &= 2(m+1)^{g-1} - 1 + (m+1) \sum_{i=2}^{2(m+1)^{g-1}} \frac{1}{\lambda_i^{g-1}} + \frac{1}{m+1} \\ &= 2(m+1)^{g-1} - 1 + (m+1)S_{g-1} + \frac{1}{m+1}. \end{aligned} \quad (22)$$

Using  $S_g^{(2)} = S_g - S_g^{(1)} = S_g - 2(m-1)(m+1)^{g-1}$  and after some simplification, Eq. (22) becomes

$$S_g = (m+1)S_{g-1} + 2m(m+1)^{g-1} - \frac{m}{m+1}. \quad (23)$$

With the initial condition  $S_0 = \frac{1}{2}$ , Eq. (23) can be solved to yield

$$S_g = \frac{(m+1)^g}{2} + (m+1)^{g-1}(2mg-1) + \frac{1}{m+1}. \quad (24)$$

Since  $\langle F \rangle_g = 2S_g$ , we have

$$\langle F \rangle_g = (m+1)^g + 2(m+1)^{g-1}(2mg-1) + \frac{2}{m+1}. \quad (25)$$

We have confirmed this closed-form expression for  $\langle F \rangle_g$  against direct computation from Eqs. (2) and (4). For all range of  $g$  and different values of  $m$ , they completely agree with each other, which shows that the analytical formula provided by Eq. (25) is right. Figure 2 shows the comparison between the numerical and predicted results, with the latter plotted by the full expression for the sum in Eq. (25).

We show next how to represent MFPT,  $\langle F \rangle_g$ , as a function of the network order  $N_g$ , with the aim to obtain the relation between these two quantities. Recalling  $N_g = 2(m+1)^g$ , we have  $(m+1)^g = \frac{N_g}{2}$  and  $g = \log_{m+1}(\frac{N_g}{2}) = \log_{m+1}N_g - \log_{m+1}2$ . These relations enable one to write  $\langle F \rangle_g$  in the following form:

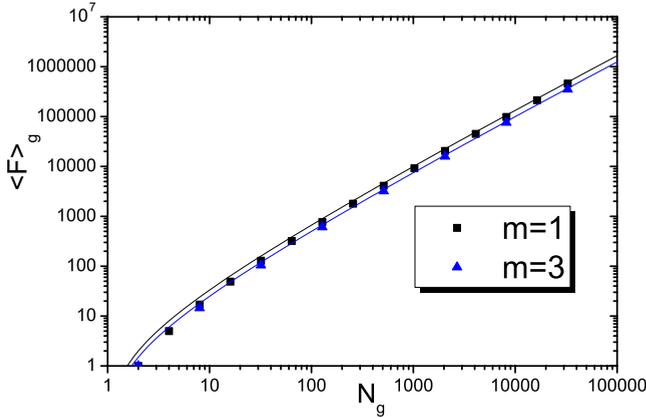


FIG. 3. (Color online) Mean first-passage time  $\langle F \rangle_g$  versus the network order  $N_g$  on a log-log scale. The filled symbols describe the analytical results shown in Eq. (26). The solid lines represent the corresponding leading scaling given by Eq. (27).

$$\langle F \rangle_g = \frac{N_g}{2} + \frac{2m}{m+1} N_g \log_{m+1} N_g - \frac{2m \log_{m+1} 2 + 1}{m+1} N_g + \frac{2}{m+1}. \quad (26)$$

Equation (26) unveils the explicit dependence relation of MFPT on the network order and parameter  $m$ . For large networks, i.e.,  $N_g \rightarrow \infty$ , we have following expression:

$$\langle F \rangle_g \sim \frac{2m}{(m+1)\ln(m+1)} N_g \ln N_g. \quad (27)$$

This leading asymptotic  $N_g \ln N_g$  dependence of MFPT with the network order is in contrast to the linear scaling previously obtained by numerical simulations for scale-free networks, such as the Apollonian networks [30] and the pseudofractal scale-free web [32]. Figure 3 shows how the MFPT scales with the network order for two values of parameter  $m$ . From Fig. 3, it is clear that for properly large network order  $N_g$ , the dominating term provided by Eq. (27) and described by the curve lines agrees well with the exact formula given by Eq. (26).

### V. MFPT FOR TRAPPING IN A SPECIAL NETWORK

In the preceding section, we have shown that the MFPT averaged over all node pairs,  $\langle F \rangle_g$ , varies with the network order  $N_g$  as  $\langle F \rangle_g \sim N_g \ln N_g$ . Below we will show that the scaling for MFPT averaged over part of node couples may be different. For this purpose, we will study the trapping issue in a particular network for  $m=1$  case, which is a random-walk problem where a trap is positioned at a given location. We focus on a special case with the trap fixed at a node with the largest degree (hereafter called hub node) absorbing all particles visiting it, which is a simplistic version of trapping in complex networks [61].

For simplicity, we continue to use the notation  $U_g$  to represent the network for  $m=1$  case after  $g$  iterations. Figure 4 illustrates the first several iterations for this network. In fact,

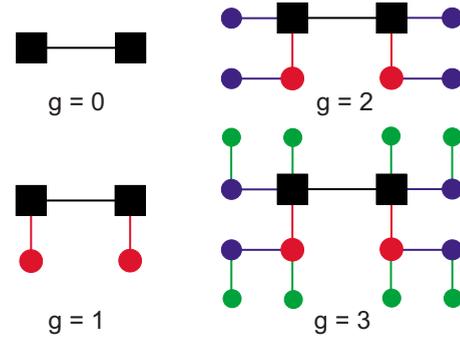


FIG. 4. (Color online) The growth process for the first three generations of the particular network corresponding to  $m=1$  case.

the network has a self-similar structure, which is obvious from the following equivalent construction method of the network: suppose one has  $U_g$ , the next generation of the network,  $U_{g+1}$ , can be obtained by joining two  $U_g$ , see Fig. 5. We call the two components,  $U_g$ , in  $U_{g+1}$  the original  $U_g$  and duplicate  $U_g$ , respectively. For the convenience of description, we label all node in  $U_{g+1}$  using the following way: the nodes in the original  $U_g$  are labeled as  $1_o, 2_o, \dots, N_{g_o}$ , while nodes in the copy of  $U_g$  are labeled as  $1_c, 2_c, \dots, N_{g_c}$ . The trap is located at the node belonging to the original  $U_g$  with label  $1_o$ .

Let  $T_i(g)$  be the trapping time (TT) of a node  $i$  in  $U_g$ , which is the expected time for a walker starting from  $i$  to first visit the trap node  $1_o$ . Obviously, for all  $g \geq 0$ ,  $T_{1_o}(g) = 0$ . We first calculate the quantity  $T_{1_c}(g)$  that is useful for deriving the main result. Since node  $1_c$  is a neighbor of the trap node  $1_o$ , according to the result obtained previously in [62], i.e., Eq. (9) in [62], we have

$$T_{1_c}(g) = N_g - 1 = 2^{g+1} - 1. \quad (28)$$

Let  $T_{\text{tot}}(g)$  denote the sum of trapping time for all nodes in  $U_g$ , i.e.,

$$T_{\text{tot}}(g) = \sum_{i \in U_g} T_i(g). \quad (29)$$

Then, the mean trapping time (MTT),  $\langle T \rangle_g$ , which is the average of  $T_i(g)$  over all initial nodes distributed uniformly in  $U_g$ , is given by

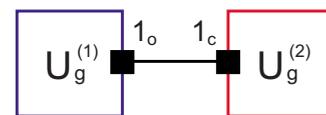


FIG. 5. (Color online) Second construction method of the network for  $m=1$  case, which highlights self-similarity. The network after  $g+1$  generations,  $U_{g+1}$ , consists of two replicas of  $U_g$  denoted separately by  $U_g^{(1)}$  and  $U_g^{(2)}$ , which are connected to each other by adding a link between two hub nodes with one in  $U_g^{(1)}$  and the other in  $U_g^{(2)}$ .

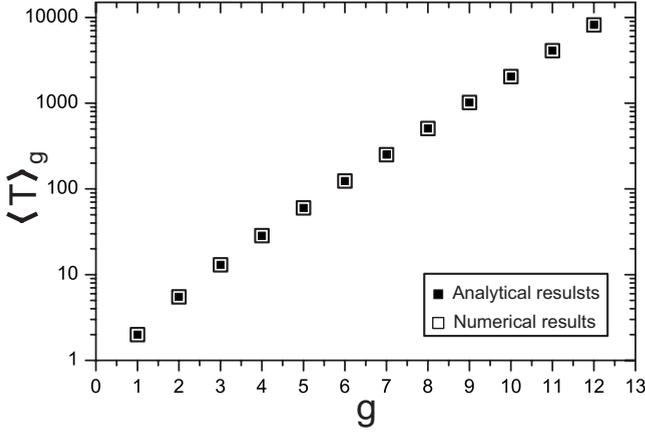


FIG. 6. Mean trapping time  $\langle T \rangle_g$  as a function of the iteration  $g$  on a semilogarithmic scale. The filled symbols are the numerical results obtained by direct calculation via Eq. (2), while the empty symbols correspond to the analytical values provided by Eq. (32). The analytical and numerical values are consistent with each other.

$$\langle T \rangle_g = \frac{1}{N_g} \sum_{i \in U_g} T_i(g) = \frac{T_{\text{tot}}(g)}{N_g}. \quad (30)$$

Thus, to obtain  $\langle T \rangle_g$ , we should first explicitly determine the quantity  $T_{\text{tot}}(g)$ , which can be settled using a recursive way.

According to the second construction method of the network, it is not difficult to express  $T_{\text{tot}}(g+1)$  in terms of  $T_{\text{tot}}(g)$ . By definition, we have

$$\begin{aligned} T_{\text{tot}}(g+1) &= \sum_{i \in U_g^{(1)}} T_i(g+1) + \sum_{i \in U_g^{(2)}} T_i(g+1) \\ &= \sum_{i \in U_g} T_i(g) + \sum_{i \in U_g^{(2)}} [F_{i_1} (g+1) + F_{i_1} (g+1)] \\ &= 2T_{\text{tot}}(g) + N_g T_{1_c}(g+1). \end{aligned} \quad (31)$$

Considering  $N_g = 2^{g+1}$ ,  $T_{1_c}(g+1) = 2^{g+2} - 1$ , and the initial condition  $T_{\text{tot}}(0) = 1$ , Eq. (31) is inductively to obtain

$$T_{\text{tot}}(g) = 4^{g+1} - 2^g(g+3). \quad (32)$$

Plugging the last expression into Eq. (30), we arrive at the closed-form expression for the MTT on network  $U_g$  for the limiting case of  $m=1$ ,

$$\langle T \rangle_g = 2^{g+1} - \frac{g+3}{2}. \quad (33)$$

Clearly, for large network (i.e.,  $N_g \rightarrow \infty$ ),

$$\langle T \rangle_g \approx N_g, \quad (34)$$

implying that the MTT  $\langle T \rangle_g$  increases linearly with the network order.

We have checked the above analytical result using extensive simulations. In Fig. 6, we plot the simulation results against Eq. (32) for different values of  $g$ . For all values of  $g$ , the numerical results are in complete agreement with the analytical results. Note that the linear dependence of MTT on network order provided by Eq. (33) is consistent with the

previously obtained results in [63] by using a simple approximate method, where it was shown, that for the trapping problem in scale-free networks having a degree distribution  $P(k) \sim k^{-\gamma}$ , when the fixed trap is positioned at a node with highest degree, the MTT  $\langle T \rangle$  varies with the network order  $N$  as  $\langle T \rangle \sim N^\beta$  with  $\beta = (\gamma - 2) / (\gamma - 1)$ . Since for an exponential network, such as the one addressed here, it can be considered as a scale-free network with  $\gamma = \infty$  [1,2], which leads to  $\beta = 1$ , in agreement with the result given in Eq. (33). Thus, the exact linear scaling obtained here confirms the general case, which was derived based on a simple continuous approximation [63].

From the above results, we know that the leading behaviors for  $\langle F \rangle_g$  and  $\langle T \rangle_g$  are evidently different. The former follows  $\langle F \rangle_g \sim N_g \ln N_g$ , while the latter obeys  $\langle T \rangle_g \sim N_g$ , less than that of the former. The distinctness between the two scalings can be interpreted by the following heuristic arguments on the basis of the peculiar structure of the network. In the trapping problem addressed here, the location for trap node is particularly selected, which lies at a node with largest degree. In fact, the trap node is the center of the network (see Fig. 4): one-half of nodes (i.e., descendants of the trap node) lies at one side of it, one-half (i.e., descendants of node  $1_c$  including  $1_c$  itself) at the other side. In this case, the walker, irrespective of its starting point, will visit at most half region of the whole network before being trapped. On the contrary, for some pairs of nodes, such as those couples of nodes with both ends being the descendants of  $1_o$  and  $1_c$ , respectively, the walker may visit large part (even the entire part) of the network before hitting the target node. This is the main reason why  $\langle T \rangle_g$  is less than  $\langle F \rangle_g$ .

Notice that, the random walks discussed in preceding section may be considered as a trapping problem with the trap uniformly distributed throughout all nodes on the networks. The different scalings between  $\langle F \rangle_g$  and  $\langle T \rangle_g$  can lead us to conclude that the location of trap has a significant effect on the leading behavior of the MTT for trapping problem on one particular network of DURTs, which is in sharp comparison with that of trapping defined on the  $T$  graph notwithstanding its tree structure, where the MTT is independent of the trap position [64]. The root of this disparity of the behaviors for random walks on a DURT and the  $T$  graph might lie in their distinct structural properties. Although they are both trees, the former is small-world with the average distance behaving logarithmically with its order [36]; while the latter is not small-world having an average distance increasing algebraically with network order [64]. Particularly, the  $T$  graph is a fractal, while the DURT is not (its fractal dimension is infinite [65]). This fractality has also been shown to distinguish diffusion in scale-free networks. For details, please see Refs. [19,28]. It should be stressed that here we only give a possible reason for this difference, the genuine explanations need further investigation in the future.

## VI. CONCLUSIONS

We have studied the standard random walks on a family of deterministic treelike networks, exhibiting small-world behavior. By applying the connection between the FPTs and

the Laplacian eigenvalues, we have determined explicitly the MFPT averaged over all pairs of nodes in the networks. The obtained solution shows that for large networks with order  $N$ , the MFPT grows approximatively with  $N$  as  $N \ln N$ . We also presented that compared to the linear scaling of looped networks, such as the Apollonian networks [30] and the pseudofractal scale-free web [32], the DURT's studied here induce a slowing down of diffusion dynamics, providing a useful insight into random walks on treelike small-world networks.

In the second part of this work, we have investigated the trapping issue on the a particular network of the DURT's, concentrating on a special case with the trapping positioned at a node with highest degree. We have obtained the explicit solution of the ATT, whose leading behavior varies lineally with network order. Based on the fact that the standard random walks addressed in the first part of the work may be looked upon as a general trapping problem with trap being distributed uniformly on every node in the whole network, we have drawn a conclusion that the scaling of ATT for

trapping depends on the location of trap. Finally, it is expected that the analytical computation methods for MFPT and ATT can be extended to other deterministic media.

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